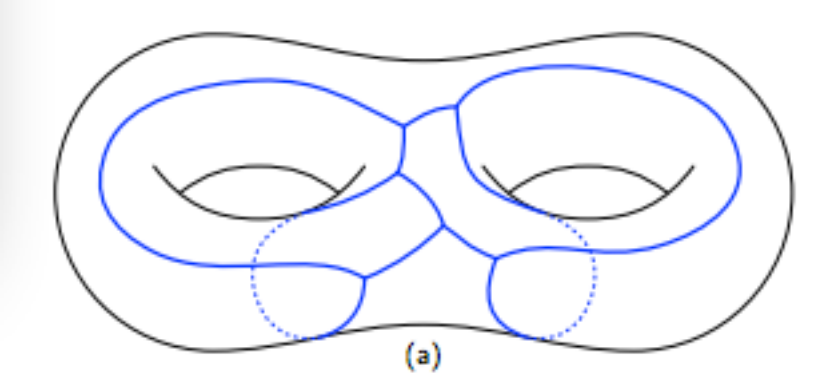
GREEDY OPTIMAL HOMOTOPY AND HOMOLOGY GENERATORS

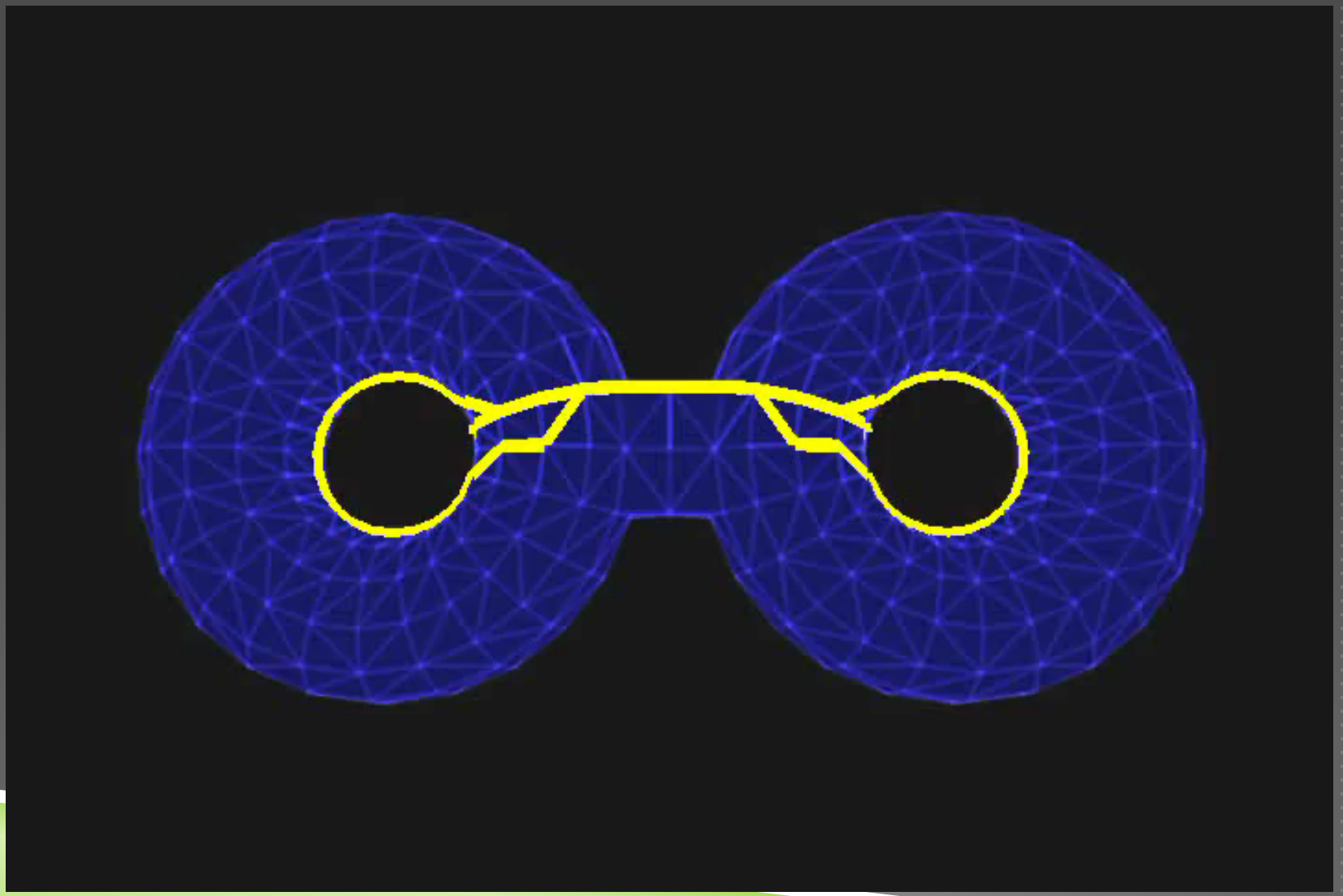
Jeff Erickson

Kim Whittlesey

MOTIVATION

- Problem: Topologically complex surfaces
  - We would much rather deal with topologically simple surfaces
- Solution: Cut the surface until it resembles a topological disk
  - Cut graph
  - What is the smallest length cut graph?
    - NP-hard!
  - What if we require the cut graph have only one vertex?
    - Greedy solution
    - As we will see, polynomial time



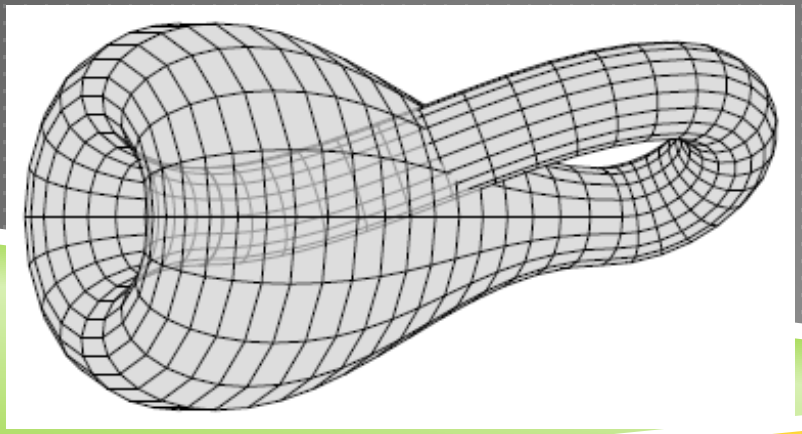


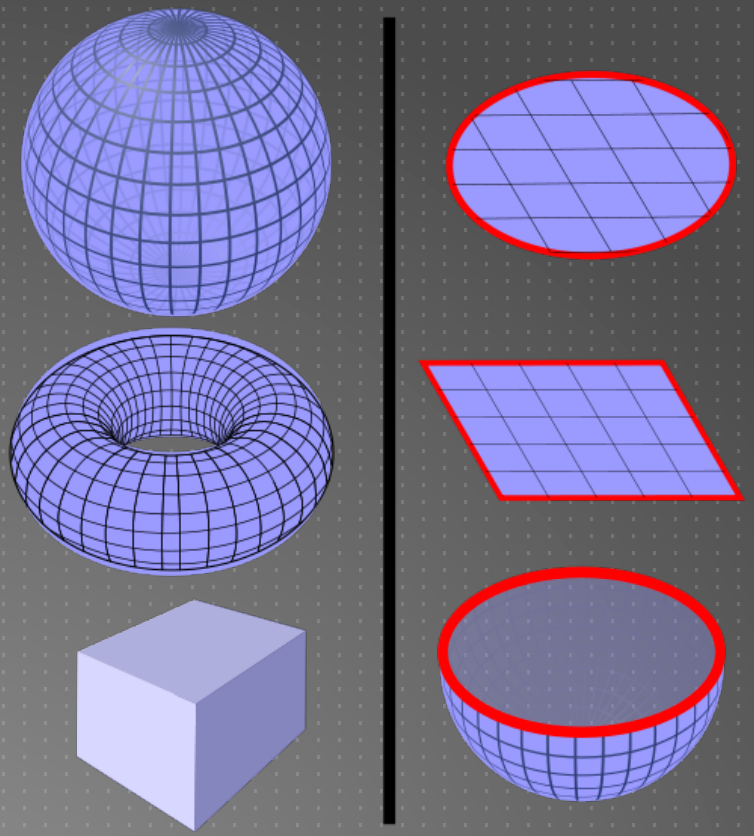
BACKGROUND

- Topology
  - 2-manifold
  - Examples
  - Combinatorial Surface
  - Cut Locus
- Algebraic Topology
  - Homotopy
  - Homology
- Graph Theory
- Mostly intuitive!

TOPOLOGY

- 2-manifold:
  - Intuitively a surface in $\mathbb{R}^3$
- Connected, compact, and orientable
  - Properties that make the surface nice
- With and without boundary
- Genus:
  - Intuitively the number of holes in the manifold



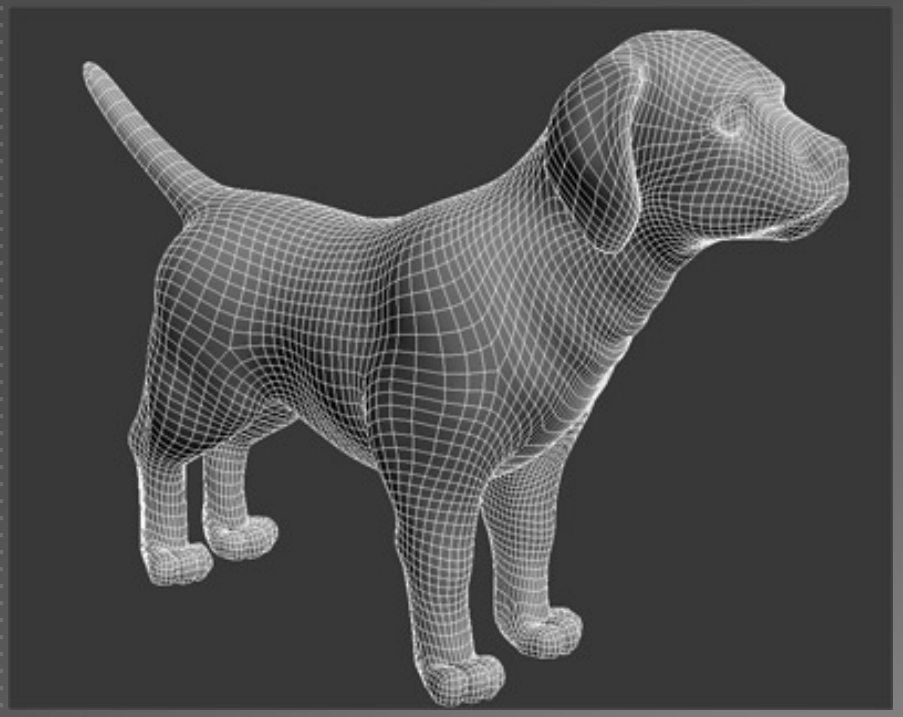


TOPOLOGY

- Examples:
  - Topological disk
    - Simplest topological space
  - 1-holed Torus – donut, coffee mug
    - Genus is 1

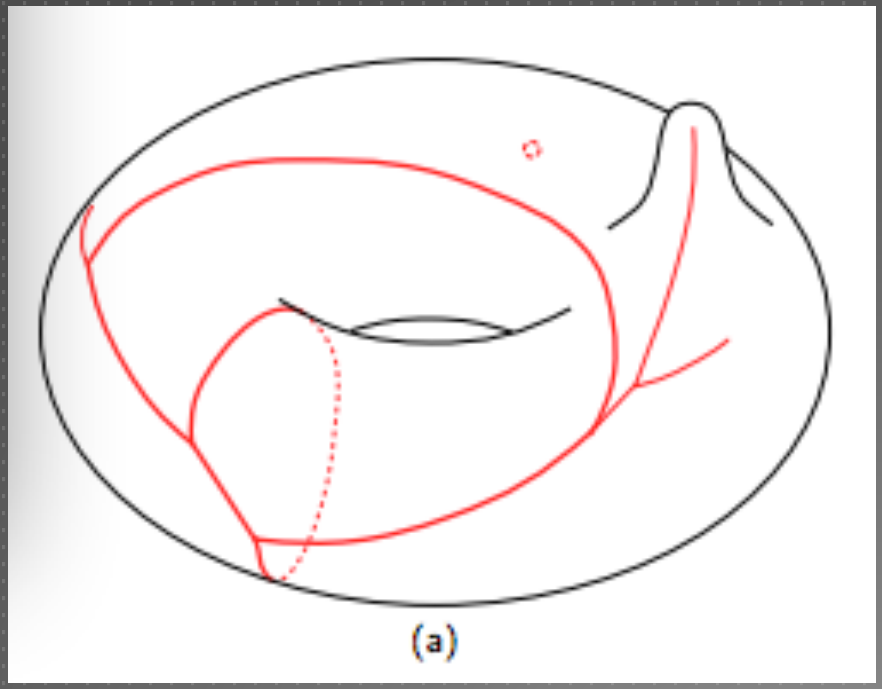


TOPOLOGY



- Combinatorial Surface $M = (M, G)$
  - Embedded graph on a 2-manifold M where every face corresponds to a topological disk
  - Edges are weighted by the lengths between vertices in M
  - Intuitively a mesh
- This puts us in a very practical context

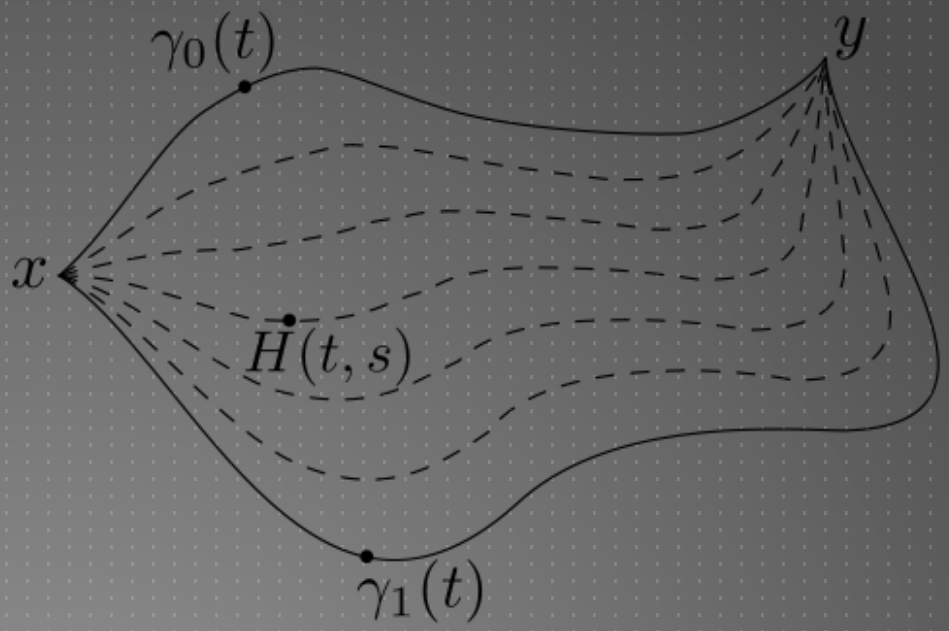
TOPOLOGY



- Cut Locus $X = X(M, x)$
  - Fix a basepoint x on M
  - X is the set of points on M with two shortest paths from x
- Reduced Cut Locus $\Phi = \Phi(M, x)$
  - Take X and remove all points where the shortest paths are homotopic
  - Let ϕ be a cut path
  - Let $\sigma(\phi)$ denote the shortest loop that crosses ϕ

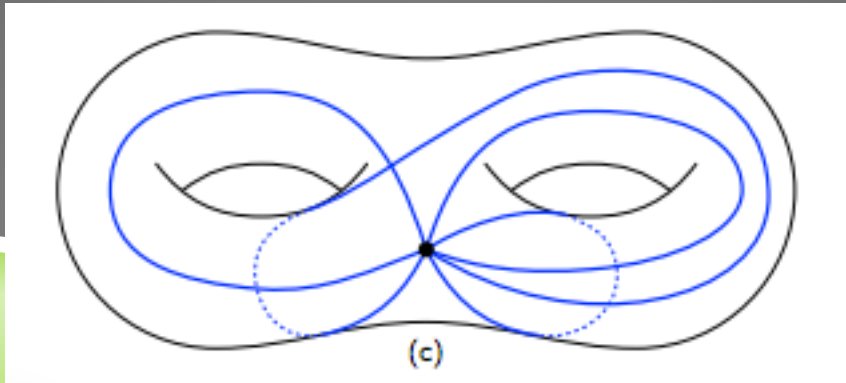
ALGEBRAIC TOPOLOGY

- Loops
  - Orientation matters
- Two loops with the same basepoint are called homotopic if there is a continuous function from one to another
  - One can be deformed until it matches the other
  - A loop homotopic to the constant loop is called contractible
- Homotopy defines equivalence classes of loops
  - Two loops are in the same class if they are homotopic



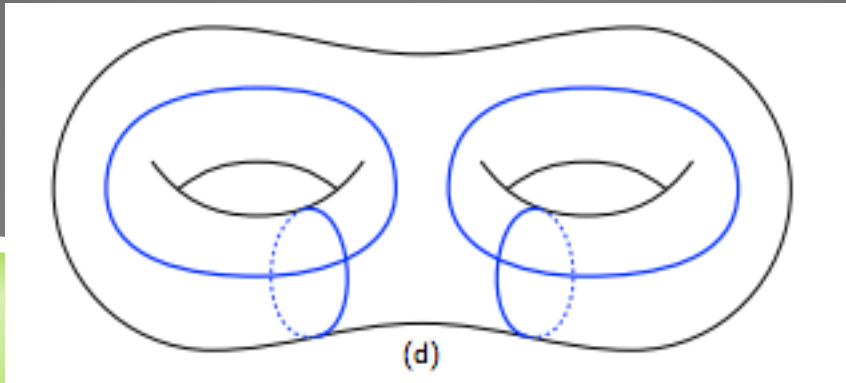
ALGEBRAIC TOPOLOGY

- The Fundamental Group $\pi_1(M, x)$
  - The equivalence classes form a group under concatenation
  - Identity element is the class of contractible loops
  - Inverse element is loop backwards
  - Unique up to isomorphism from chosen x
  - Invariant of the manifold
- Homotopy Basis:
  - Any set of 2g (g is the genus) loops that generate the fundamental group



ALGEBRAIC TOPOLOGY

- Homology:
  - Also equivalence classes of cycles
  - Two loops are in the same homology class if their difference is a surface
  - The identity element in this case is the class of *separating cycles*
- Homology Basis:
  - Any set of 2g loops that generate $H_1(M;R)$ isomorphic to R^{2g}
  - Any homotopy basis is a homology basis, but not vice versa
    - Homology bases do not have to have a common basepoint



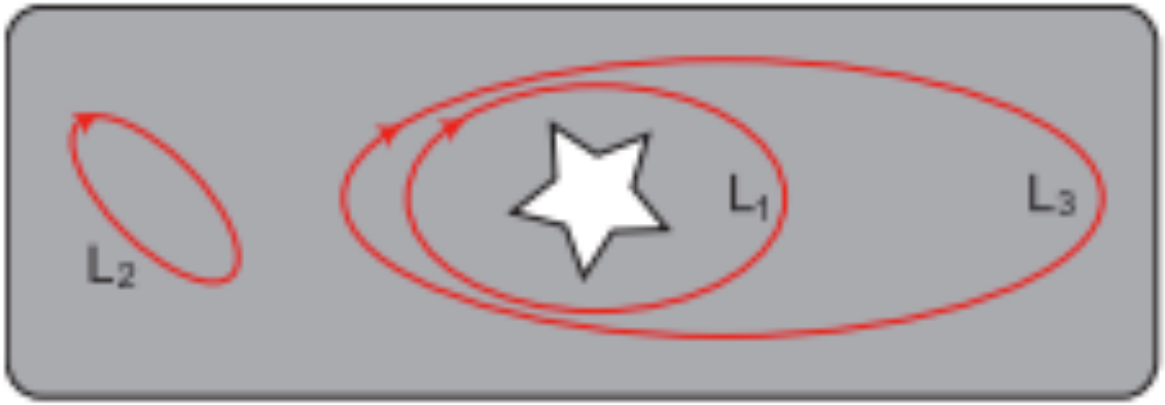


Figure 11: *Example of Homology Classes: the cycles L_1 and L_2 are topologically distinct as one encloses a hole while the other does not; L_1 and L_3 are however in the same equivalence class.*

THE GREEDY HOMOTOPY BASIS

- Given an oriented 2-manifold M (a smooth surface) and basepoint x
- Define greedy homotopy basis $\gamma_1, \gamma_2, \cdots, \gamma_{2g}$ as follows:

> For each i, γ_i is the shortest loop ℓ such that $M \setminus (\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_{i-1} \cup \ell)$ is connected.

- We will show this is the shortest set of generators of the fundamental group

LEMMA 1

- Lemma: Every loop in the greedy homotopy basis has the form $\sigma(\phi)$ for some cut path ϕ
- Proof:
  - Imagine a circular wavefront growing from the basepoint, x
  - Sometimes, the wavefront meets itself
  - This corresponds to a loop and by definition is on the cut locus

LEMMA 1

- We can replace a loop l with $\sigma(\phi)$
- Since the greedy loops generate all loops, we can write any loop as a linear combination of the greedy loops
  - $[l]$ denotes the homology class of l

$$[l] = \sum_{i=1}^{2g} \lambda_i [\gamma_i]$$

  - If λ_i is non-zero, we call it a greedy factor of l

LEMMA 2

- Lemma: If γ is a greedy factor of $\sigma(\phi)$, then $|\gamma| \leq |\sigma(\phi)|$
- Proof:
  - Let $\gamma_1, \ldots, \gamma_i$ be all the greedy loops shorter than $\sigma(\phi)$
  - If $\sigma(\phi)$ is the next greedy loop, it is its only own greedy factor
  - If $\sigma(\phi)$ instead disconnects M, it is in the homology class of a subset of the greedy loops
  - Therefore, its greedy factors have length less than $|\sigma(\phi)|$

LEMMA 3

- Lemma: If γ is a greedy factor of an arbitrary loop l, then $|\gamma| \leq |l|$
- Proof:
  - Suppose l crosses the reduced cut locus Φ
  - It then crosses some set of cut paths $\phi_1, \phi_2, \ldots, \phi_r$
  - By definition, l is longer than the concatenation $\sigma(\phi_1)\sigma(\phi_2)\ldots\sigma(\phi_r)$

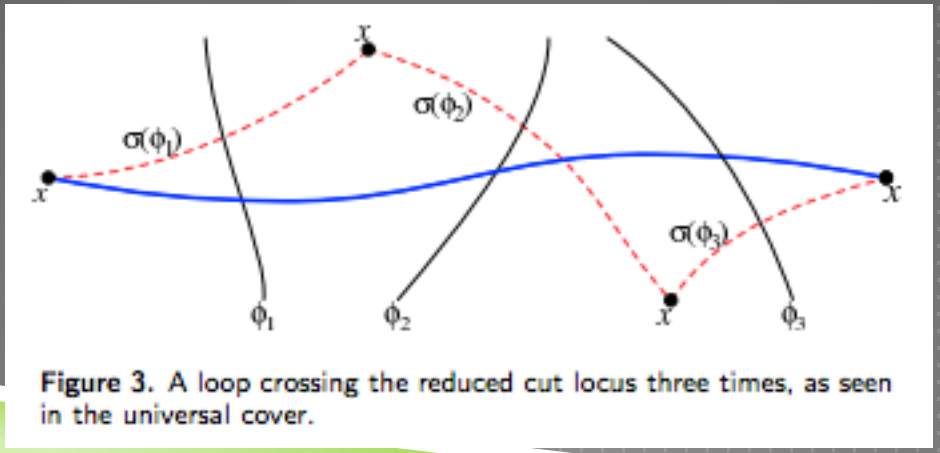


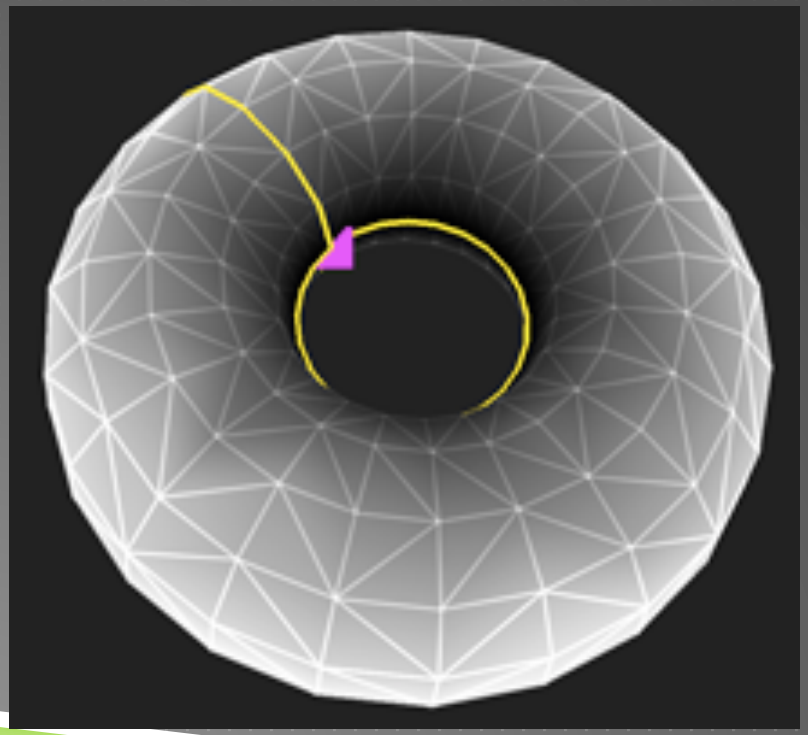
Figure 3. A loop crossing the reduced cut locus three times, as seen in the universal cover.

LEMMA 4

- Lemma: Let $\{\alpha_1, \alpha_2, \ldots, \alpha_{2g}\}$ be any set of loops that generate the fundamental group $\pi_1(M, x)$. There is a permutation $\pi \in S_{2g}$ such that for each i, $\gamma_{\pi(i)}$ is a greedy factor of α_i
- Proof:
  - Notice that there is a nonsingular linear transformation from one basis to the other
  - Then it has nonzero determinant
  - This implies the existence of an appropriate permutation

MAIN THEOREM

- Theorem: For any 2-manifold M and any basepoint x in M, the greedy homotopy basis is the shortest set of generators of $\pi_1(M, x)$.
- Proof:
  - Directly follows from Lemmas 3 and 4



LEMMA 5

- Lemma: Each loop γ_i in the greedy homotopy basis is the shortest loop in its homotopy class
- Proof:
  - Suppose ϕ^* is the shortest loop in the homotopy class of $\sigma(\phi)$ and $\phi^* \neq \sigma(\phi)$
  - Then every greedy factor of $\sigma(\phi)$ is strictly shorter than $\sigma(\phi)$
  - This implies $\sigma(\phi)$ is not a greedy loop

SMOOTH SETTING

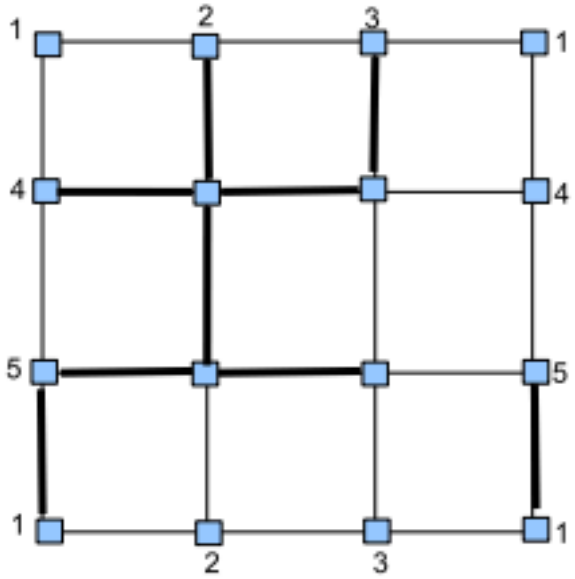
1. Compute the reduced cut locus Φ
2. Weight each cut path ϕ with the length of its shortest crossing loop $\sigma(\phi)$
3. Order the cut paths by increasing weight
4. If $\Phi \setminus \phi$ is connected, remove ϕ from Φ and declare $\sigma(\phi)$ to be the next greedy loop

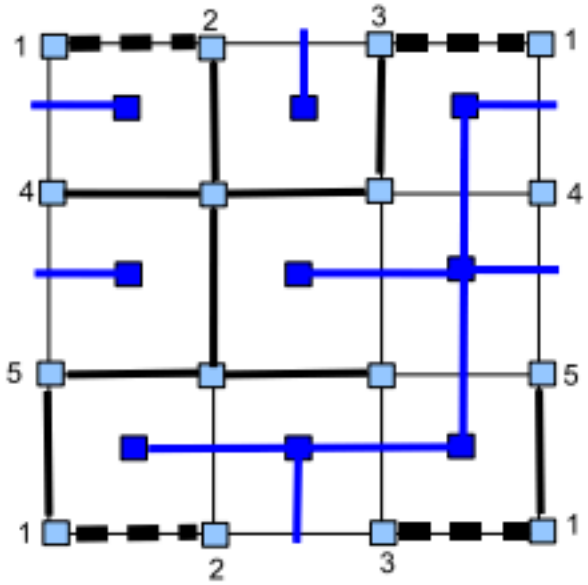
DISCRETE SETTING

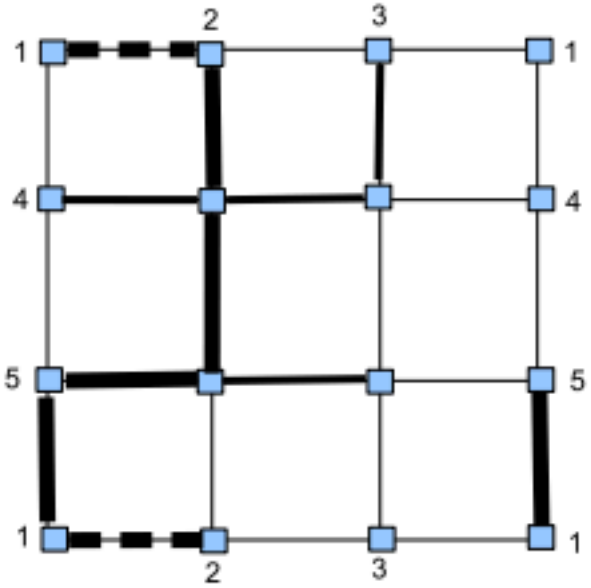
- Notice that we can also use the cut locus's maximum spanning tree to find the greedy loops
- Use mesh as combinatorial surface
- Consider tree-cotree decomposition
  - Eppstein, 2003

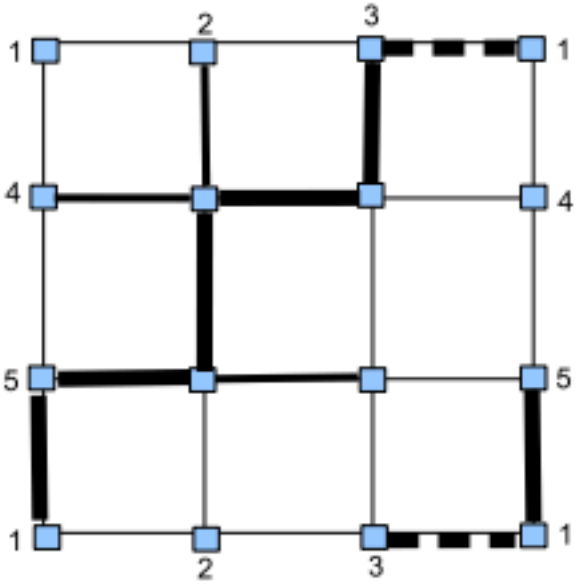
TREE-COTREE ALGORITHM

1. Build a spanning tree T of primal edges.
2. Build a spanning tree T^* of dual edges that do not cross edges in T.
3. For each dual edge e^*_{ij} that is neither contained in T^* nor crossed by T, follow both of its endpoints back to the root of T. The resulting loop is a generator.


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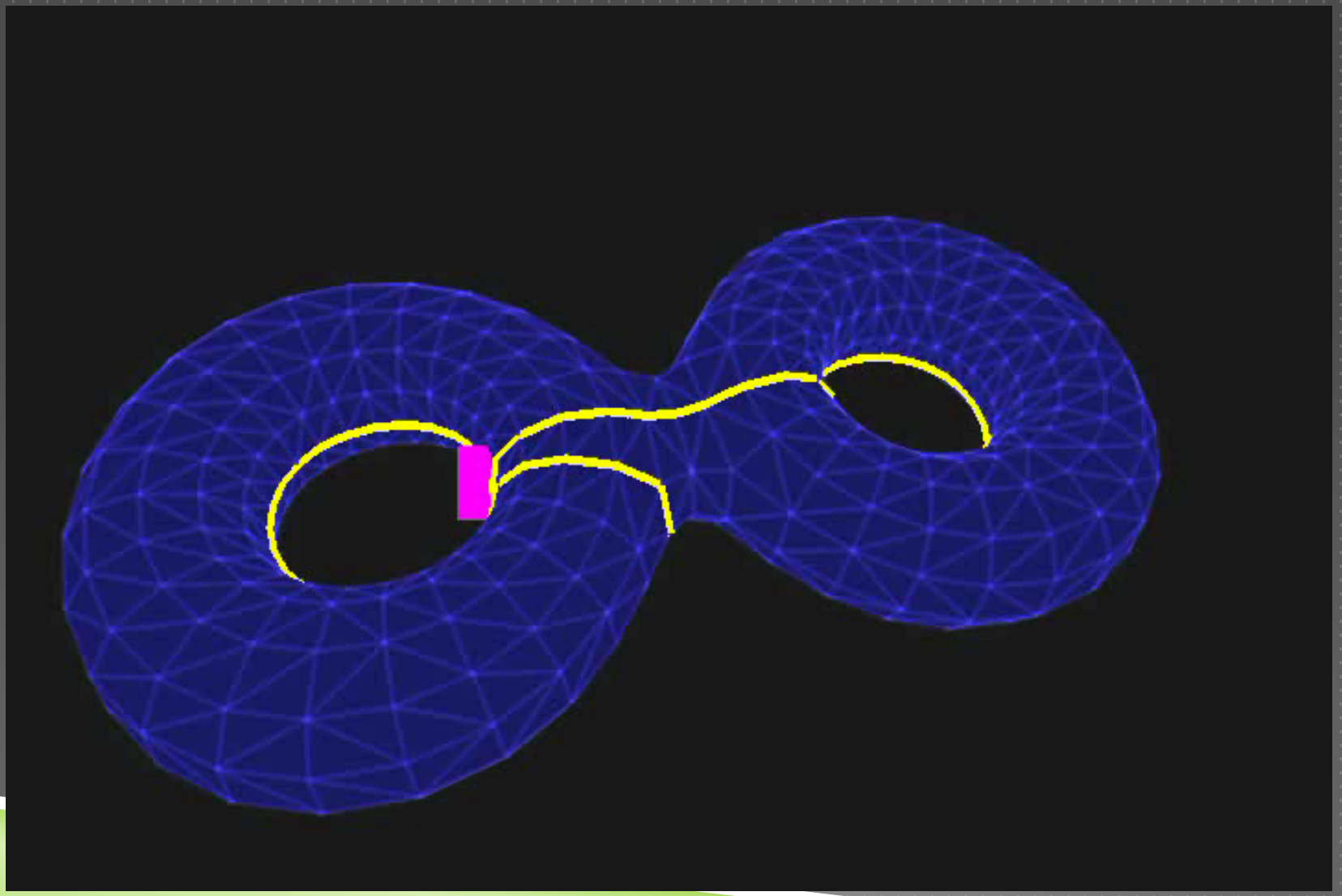

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TREE-COTREE ALGORITHM

- Exactly 2g edges remaining
  - Euler's Formula: $V - E + F = 2(1 - g)$
- These necessarily form a set of generators

DISCRETE SETTING

- Notice that we can also use the cut locus's maximum spanning tree to find the greedy loops
- Use mesh as combinatorial surface
- Consider tree-cotree decomposition
  1. Find minimum paths tree
  2. Weight remaining edges e with $\sigma(e)$, the shortest loop containing e
  3. Find maximum spanning cotree
  4. The remaining loops give the greedy homotopy basis
  5. Repeat for every basepoint x



COMPLEXITY

- Shortest paths tree
  - Can use Dijkstra's algorithm!

    $O(n \log(n))$
- Computing each $\sigma(e)$ is $O(n)$
- Maximum dual spanning tree
  - Can use Prim's or Kruskal's algorithm
  - $O(n \log(n))$
- Over all basepoints, total run time is $O(n^2 \log(n))$

GREEDY HOMOLOGY BASIS

- Not necessarily a cut graph
  - Applications elsewhere
- We are interested in $H_1(M;R) \cong R^{2g}$
  - Let $R = Z_2$
  - This makes R^{2g} a vector space
    - Optimality of greedy algorithm follows
    - At each step, choose shortest cycle that is not a linear combination of previously chosen cycles

LEMMAS 6 AND 7

- Lemma: Every cycle in the shortest homology basis is tight
  - Proof: Otherwise, we could decompose the cycle into a shorter homology basis
- Lemma: Every cycle in the shortest homology basis has the form $\sigma(x, \phi)$ for some cut path ϕ in $\Phi(x)$
  - Proof: Every tight loop is of this form

HOMOLOGY BASIS ALGORITHM

- Fix a basepoint x
- Compute the reduced cut locus $\Phi(x)$
- Compute the greedy *homotopy* basis (necessarily also a homology basis)
- For each edge e, calculate $[\sigma(x, e)]$ as a linear combination of basis elements
  - Now we have a bit vector for every edge in M

HOMOLOGY BASIS ALGORITHM

- For each basepoint y
  - Compute $\Phi(y)$ and every $\sigma(y, \phi)$
  - Compute the homology class for every $\sigma(y, \phi)$
- Order the σ loops by increasing length
- Choose the next loop that is independent from previous basis loops
- Runtime is $O(n^2 \log(n) + n^2 g + n g^3)$

WITH BOUNDARY

- Extension by Dlotko, 2011
- Similar to case without boundary
- Modify tree-cotree algorithm:
  1. Find a dual spanning tree
  2. Add to this tree a single edge dual to a boundary edge in M
  3. Find a primal spanning tree
  4. The remaining edges determine loops which give the set of generators

